DIFFERENTIAL EQUATIONS

HOMOGENEOUS FORM
$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

SUBSTITUTE $y = vx$

e.g. $\frac{dy}{dx} = \frac{x+2y}{x} = 1 + 2\left(\frac{y}{x}\right)$

LET $y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$\therefore x\frac{dv}{dx} + v = 1 + 2\left(\frac{vx}{x}\right)$

$x\frac{dv}{dx} = 1 + v$

$\int \frac{1}{1+v}\,dv = \int \frac{1}{x}\,dx$

$\ln|1+v| = \ln|x| + c$

$1 + \frac{y}{x} = Ax$

$y = Ax^2 - x$

INTEGRATING FACTOR
$$\frac{dy}{dx} + P(x)y = Q(x)$$

MULTIPLY THROUGH BY... $I(x) = e^{\int P(x)dx}$

e.g. $\frac{dy}{dx} + 2y = e^x$

$\therefore I(x) = e^{\int 2dx} = e^{2x}$

$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{3x}$

$\frac{d}{dx}(y \cdot e^{2x}) = e^{3x}$

$ye^{2x} = \int e^{3x}dx$

$ye^{2x} = \frac{1}{3}e^{3x} + c$

$y = \frac{e^x}{3} + \frac{c}{e^{2x}}$

VARIABLES SEPARABLE
$$\frac{dy}{dx} = f(x).g(y)$$

SEPARATE AND INTEGRATE $\int \frac{1}{g(y)}dy = \int f(x)dx$

e.g. $\frac{dy}{dx} = xy - x = x(y-1)$

$\therefore \int \frac{1}{y-1}dy = \int x\,dx$

$\ln|y-1| = \frac{1}{2}x^2 + c$

$y - 1 = e^{\frac{1}{2}x^2} \cdot e^c$

$y - 1 = Ae^{\frac{1}{2}x^2}$

$y = Ae^{\frac{1}{2}x^2} + 1$

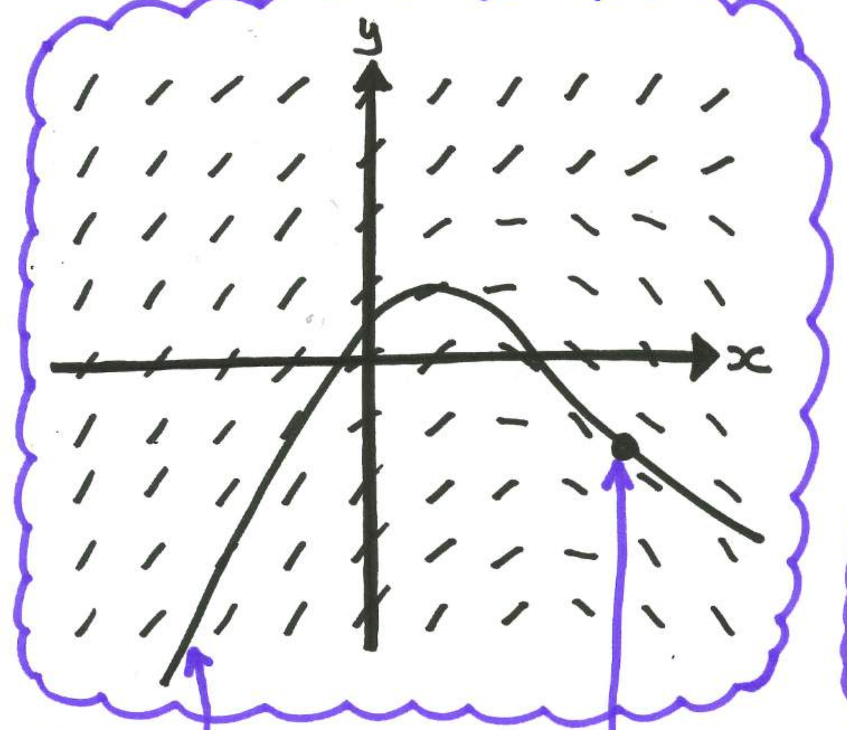
RATES OF CHANGE
e.g. POPULATION GROWTH, CARBON DATING, LAW OF COOLING, etc...

GENERAL SOLUTION
$\downarrow (x_0, y_0)$

PARTICULAR SOLUTION

SLOPE FIELD
JOINS POINTS OF EQUAL GRADIENT

ISOCLINES



SOLUTION CURVE

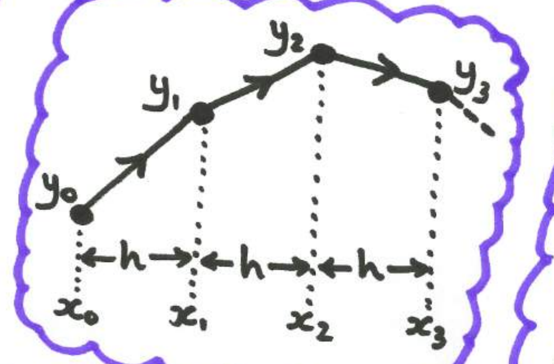
BOUNDARY CONDITION

NUMERICAL METHODS
$$\frac{dy}{dx} = f(x,y)$$

EULER'S METHOD FOR CONSTANT STEP h

$y_{n+1} = y_n + h f(x_n, y_n)$

$x_{n+1} = x_n + h$



TAYLOR SERIES
e.g. $\frac{dy}{dx} = x^2 + y^2,\ x=1, y=0$

$\Rightarrow y(1) = 0$

$\Rightarrow y'(1) = 1^2 + 0^2 = 1$

$y'' = 2x + 2yy'$

$\Rightarrow y''(1) = 2 + 2.0.1 = 2$

$y''' = 2 + 2yy'' + 2y'y'$

$\Rightarrow y'''(1) = 2 + 2.0.2 + 2.1.1 = 4$

TAYLOR SERIES

$y = y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2!} + \ldots$

$\therefore y \approx (x-1) + (x-1)^2 + \frac{2}{3}(x-1)^3 + \ldots$

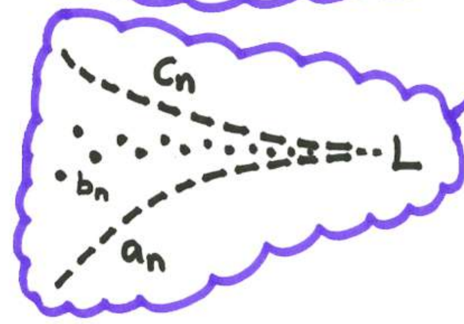
SO $y(1.1) \approx 0.1107$

LIMITS

SQUEEZE THEOREM
IF $a_n \le b_n \le c_n\ \forall n \in \mathbb{Z}^+$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L < \infty$

THEN $\lim_{n\to\infty} b_n = L$



L'HÔPITAL'S RULE
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

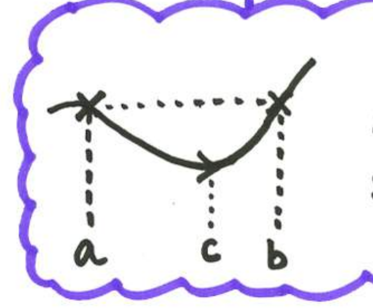
"$\frac{0}{0}$" AND "$\frac{\infty}{\infty}$"

e.g. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \frac{1}{1} = 1$

CONTINUOUS AT $x = a$ IF
$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$

DIFFERENTIABLE AT $x = a$ IF
$\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h}$ AND $\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$ EXIST AND ARE EQUAL.

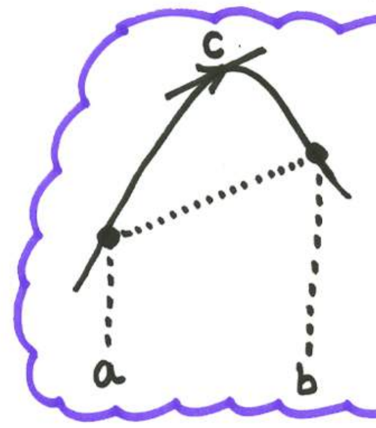
FOR A FUNCTION $f(x)$ WHICH IS
- CONTINUOUS ON $[a,b]$
- DIFFERENTIABLE ON $]a,b[$ ....

ROLLE'S THEOREM


IF $f(a) = f(b)$

$\exists\ c \in\ ]a,b[$

s.t. $f'(c) = 0$

MEAN VALUE THEOREM


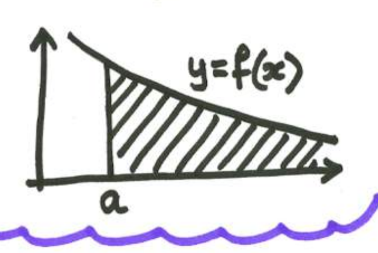
$\exists\ c \in\ ]a,b[$ SUCH THAT

$f'(c) = \frac{f(b)-f(a)}{b-a}$

IMPROPER INTEGRALS

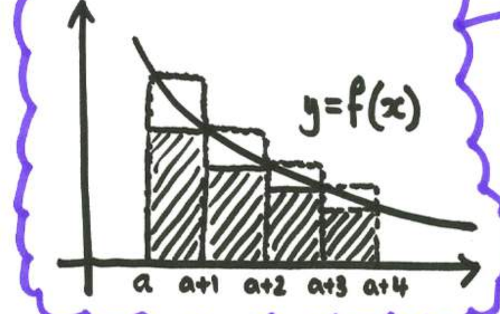
FUNDAMENTAL THEOREM OF CALCULUS
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$



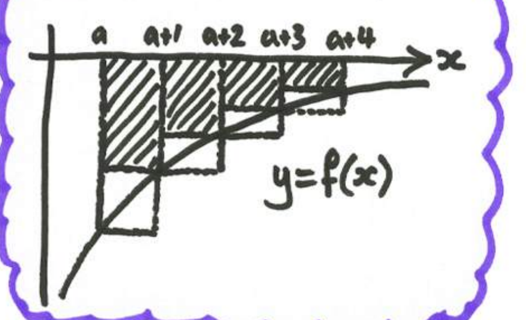
RIEMANN SUMS

$f(x)$ DECREASING



$$\sum_{k=a+1}^{\infty} f(k) < \int_a^\infty f(x)dx < \sum_{k=a}^{\infty} f(k)$$

$f(x)$ INCREASING



$$\sum_{k=a}^{\infty} f(k) < \int_a^\infty f(x)dx < \sum_{k=a+1}^{\infty} f(k)$$

INFINITE SERIES

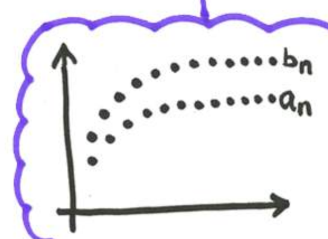
P-SERIES
$\sum_{k=1}^{\infty}\frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$

$p = 1 \Rightarrow$ HARMONIC SERIES

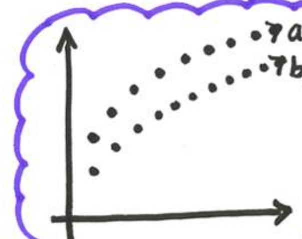
$p \le 1 \Rightarrow$ DIVERGES

$p > 1 \Rightarrow$ CONVERGES

COMPARISON TEST
$\sum b_n$ CONVERGES $\Rightarrow \sum a_n$ CONVERGES



$\sum b_n$ DIVERGES $\Rightarrow \sum a_n$ DIVERGES



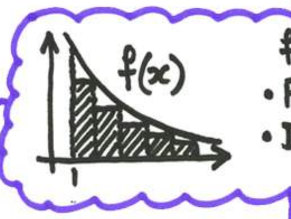
LIMIT COMPARISON TEST
$\lim_{k\to\infty}\frac{a_k}{b_k} = c,\quad 0 < c < \infty$

$\sum a_n$ CONVERGES $\Leftrightarrow \sum b_n$ CONVERGES

$\sum a_n$ DIVERGES $\Leftrightarrow \sum b_n$ DIVERGES

DIVERGENCE TEST
$\lim_{k\to\infty} u_k \ne 0$

$\Rightarrow \sum u_k$ DIVERGES

INTEGRAL TEST


$f(x)$ IS
- POSITIVE
- DECREASING

$\int_1^\infty f(x)dx$ CONVERGES $\Rightarrow \sum f(k)$ CONVERGES

$\int_1^\infty f(x)dx$ DIVERGES $\Rightarrow \sum f(k)$ DIVERGES

RATIO TEST
$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = c$

$c < 1 \Rightarrow \sum u_n$ CONVERGES

$c > 1 \Rightarrow \sum u_n$ DIVERGES

$c = 1 \Rightarrow$ NO CONCLUSION

POWER SERIES
$\sum_{k=0}^{\infty} a_k(x-b)^k = a_0 + a_1(x-b) + a_2(x-b)^2 + \ldots$

CONVERGES FOR $|x-b| < R$

$b - R < x < b + R$

R IS THE RADIUS OF CONVERGENCE

CHECK $x = b - R$, $x = b + R$

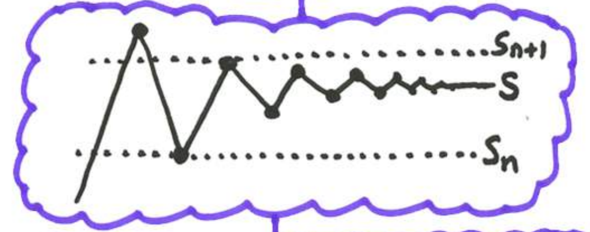
THE INTERVAL OF CONVERGENCE

$R = 0$ CONVERGES ONLY FOR $x = b$

$R = \infty$ CONVERGES $\forall x \in \mathbb{R}$

ALTERNATING SERIES TEST
$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \ldots$

CONVERGES IF $|u_{n+1}| < |u_n|$ AND $\lim_{n\to\infty}|u_n| = 0$



TRUNCATION ERROR $= |S - S_n| < |u_{n+1}|$

eg. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ CONVERGES

BUT $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ DIVERGES

CONDITIONALLY CONVERGENT

ABSOLUTELY CONVERGENT

eg. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \ldots$ CONVERGES

AND $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ CONVERGES

MACLAURIN AND TAYLOR SERIES

MACLAURIN SERIES – 'CENTRED AT 0'
$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

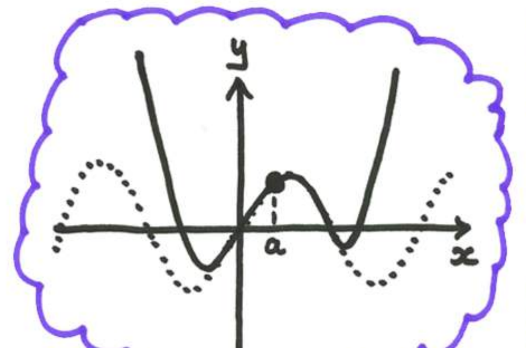
$f''(0) = -\cos(0) = -1$

$f'''(0) = \sin(0) = 0$

$f''''(0) = \cos(0) = 1$

etc...

e.g. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$



TAYLOR SERIES – 'CENTRED AT a'
$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$

TAYLOR APPROX.

$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x)$

LAGRANGE FORM FOR THE ERROR

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ WHERE c LIES BETWEEN a AND x

CHOOSE c TO MAXIMISE $f^{(n+1)}(c)$ TO FIND UPPER BOUND.

FIND LIMITS
e.g. $\lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

$= \lim_{x\to 0}\frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots - 1 + \frac{x^2}{2}}{x^4}$

$= \lim_{x\to 0}\frac{1}{4!} + \frac{x^2}{6!} + \ldots$

$= \frac{1}{4!}$

FIND INTEGRALS
e.g. $\int_0^1 e^{-x^2}dx$

$= \int_0^1 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\, dx$

$= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \ldots\right]_0^1$

$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots \approx \frac{26}{35}$

UWC SOUTH EAST ASIA